Option Prices under Generalized Pricing Kernels

BERTRAM DÜRING Duering@uni-mainz.de
Institut für Mathematik, J. Gutenberg-Universität Mainz, 55099 Mainz, Germany

ERIK LÜDERS* Erik.Lueders@fas.ulaval.ca
Département de finance et assurance, Université Laval, Quebec G1K7P4, Canada

Abstract. In this paper analytical solutions for European option prices are derived for a class of rather general asset specific pricing kernels (ASPKs) and distributions of the underlying asset. Special cases include underlying assets that are lognormally or log-gamma distributed at expiration date T. These special cases are generalizations of the Black and Scholes (1973) option pricing formula and the Heston (1993) option pricing formula for non-constant elasticity of the ASPK. Analytical solutions for a normally distributed and a uniformly distributed underlying are also derived for the class of general ASPKs. The shape of the implied volatility is analyzed to provide further understanding of the relationship between the shape of the ASPK, the underlying subjective distribution and option prices. The properties of this class of ASPKs are also compared to approaches used in previous empirical studies.

Keywords: Pricing kernel, option pricing, partial differential equation, finite differences, implied volatility.

JEL classification: G12, G13, C65

Introduction

Independently of the approach chosen to price options, as with any pricing problem, option prices are completely determined by the distribution of the value of the underlying asset at expiration and the shape of the asset specific pricing kernel (ASPK).[1] Mathematically, the pricing kernel characterizes the change from the subjective probability measure P to the risk-neutral (or equivalent martingale) measure Q. It is also known as the Radon-Nikodym derivative of Q with respect to P. In the case of Black and Scholes (1973) it is assumed that the underlying asset is governed by a geometric Brownian motion and that continuous and frictionless trading is possible. For the Black and Scholes model, both the distribution at expiration and the ASPK are uniquely determined by the geometric Brownian motion. Rubinstein (1976) and Brennan (1979) make explicit assumptions on the distribution and the ASPK. More precisely, they assume a representative investor and thus the representative investor's utility function characterizes the ASPK. Câmara (2003) and Schroder (2004) recently extended their approach to alternative distributions and utility functions. However, Câmara (2003) and Schroder (2004) also focus on preferences and distributions which yield risk neutral valuation relationships, i.e. option pricing formulas without any preference parameter.

In contrast to these models, in this paper we do not restrict our analysis to such risk neutral valuation relationships. The focus of this paper is to derive analytical option pricing formulas which impose as little as possible restrictions on the shape

* Erik Lüders is an assistant professor at Laval University and a visiting scholar at the Stern School of Business, New York University.

of the ASPK and the distribution of the underlying asset. The option pricing formulas are based on a generalized polynomial characterization of the ASPK. Based on this general characterization of ASPKs, we derive analytical option pricing formulas for lognormally, log-gamma, uniformly and normally distributed underlyings. We also demonstrate how the approach works in general and we point out that our approach works with many alternative distributions. Although analytical option pricing formulas exist for these distributions, our approach is based on much weaker assumptions on the ASPK. For example, Heston (1993) derives an analytical option pricing formula for a European option on a log-gamma distributed underlying if the ASPK has constant elasticity. We derive an analytical option pricing formula on a log-gamma distributed underlying for any ASPK that can be characterized by a generalized polynomial.

While for many underlyings as, for example, stocks and stock market indices the lognormal or the log-gamma distribution might be considered as good approximations there is no reason to assume that the ASPK should be characterized by constant elasticity as in the Black-Scholes or the Heston models. Even if constant relative risk aversion might be a good model for the representative investor and thus the market pricing kernel has constant elasticity, this does not imply that the ASPK should have constant elasticity with respect to the underlying.[2] Indeed, recent empirical studies by Ait-Sahalia and Lo (2000), Jackwerth (2000) and Rosenberg and Engle (2002) suggest that even the market pricing kernel is not of the constant elasticity type. Several papers have analyzed the impact of non-constant elasticity of the ASPK on option prices (see for example Benninga and Mayshar, 2000, and Franke, Stapleton and Subrahmanyam, 1999)[3]. However, they did not provide an analytical option pricing formula. Thus, our analytical option pricing model is a valuable extension of the existing option pricing literature. Its main advantages are:

First, the enhanced flexibility allows for more accurate analytical option pricing formulas. For example, our generalized Heston-model which is based on the log-gamma distribution seems to be a promising approach for pricing options on a broad-based stock index as the S&P 500. The log-gamma distribution is relatively flexible and allows to fit reasonable levels of kurtosis and skewness and the generalized polynomial characterization of the ASPK is flexible enough to fit empirical ASPKs as will be shown.

Second, our approach provides a convenient way to analyze the quantitative implications of non-constant elasticity of the ASPK on option prices.

Third, the flexibility in terms of the distribution as well as in terms of the ASPK provides also a new tool to infer ASPKs from option data. Compared to existing parametric methods (see, for example, Bartunek and Chowdhury, 1997), our approach has the advantage of enhanced flexibility while still yielding a parametric estimate of the ASPK.

Fourth, our approach does not rely on complete markets. Since only the expected terminal distribution of the underlying needs to be known it presents an interesting model to price options when the underlying is not traded as, for example, with real options. We discuss the applicability of our formulas to real options and employee stock options as well as the economic intuition behind using a polynomial approximation of the ASPK in both cases.

This paper is closely related to the companion paper Lüders and Franke (2004). The polynomial characterization of the ASPK was originally proposed by Lüders and Franke (2004) to derive analytical time-series models for the market portfolio. While Lüders and Franke analyze the time-series characteristics of the market portfolio, this paper focuses on option prices. Also, Lüders and Franke (2004) is an analysis in continuous time. This paper also extends Lüders and Franke (2004) since they consider only the lognormal case. Moreover, we analyze in greater detail the characteristics of the new ASPK class and we show that the polynomial characterization can be used to infer empirical ASPK. Finally, in this paper empirical ASPKs estimated in Jackwerth (2000) are fitted.

The paper is organized as follows. Section 1 presents the market model and the class of generalized ASPKs which allows for very general shapes of the ASPK to be matched. The characteristics of the polynomial ASPKs are also analyzed. In Section 2 the general approach to value options with generalized polynomial approximations of the ASPK is shown. Based on this class of ASPKs, we derive an analytical pricing formula for European options, when the final distribution at time T is lognormal and the option matures at time T. In order to price options which mature at time $\tau < T$ and to characterize the influence of the ASPK on the price and the implied volatilities, we derive a generalized Black-Scholes partial differential equation for the option price in Section 2.2.2. We solve this equation numerically for a specific version of the general ASPK proposed above using a standard finite difference scheme. We then turn to the cases when the underlying is log-gamma, uniformly and normally distributed and derive analytical option pricing formulas for these cases. We compare the different models in terms of implied volatilities. The paper is completed by a short conclusion.

1. The model

Throughout this paper we consider a market with a given time horizon $T > 0$. The different examples in this paper will differ with respect to the information structure, i.e. the filtrations will vary. We assume that the asset does not pay any dividends until terminal date T. The fundamental asset pricing equation states that in an arbitrage free market the price of an asset is given by the expected future value of the asset, where the expectation is taken under some equivalent martingale measure Q. To simplify the presentation in this paper we always assume the risk-free rate to be zero.[4] The equivalent martingale measure Q is defined by

$$Q(A) = \int_A \phi_{t,T} \, dP, \quad \forall A \in \mathcal{F}_T,$$

with the physical measure P and the asset specific pricing kernel $\phi_{t,T}$. Given the risk-free rate is zero and given the equivalent martingale measure is defined by the ASPK $\phi_{t,T}$, the asset price F_t for $0 \le t \le T$ can be written as[5]

$$F_t = \mathrm{E}^Q(I_T) = \mathrm{E}\left(I_T \phi_{t,T} \middle| \mathcal{F}_t \right),$$

where I_T is the value of an information process at the terminal date T and the filtration $\mathcal{F}_t$ characterizes the information available at time $0 \le t \le T$. The information process is exogenously given and defined as the conditional expectation of the terminal value of the underlying asset. Due to the definition of the information process I_t, the value I_T is equal to the terminal time T value of the underlying. This may be either some liquidation value at time T or simply the asset price of the underlying at time T.[6] It follows that the price of a European call option with strike price K and expiration date T is given by

$$C_t = \mathrm{E}\left(\max(I_T - K, 0)\, \phi_{t,T} \middle| \mathcal{F}_t\right), \quad 0 \le t \le T.$$

Throughout this paper, we will assume that the distribution of I_T and the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ are exogenously given. Our emphasis is to analyze the impact of the ASPK on European option prices. In order to get analytical option pricing formulas we follow Lüders and Franke (2004) who characterize the ASPK by a generalized polynomial. In the following subsections we present the polynomial characterization and discuss its properties. Analytical formulas for European option prices are then derived for alternative distributional assumptions of the underlying asset.

1.1. A general characterization of asset specific pricing kernels

Lüders and Franke (2004) suggest to characterize the ASPK by a generalized polynomial, i.e.

$$\phi_{t,T} = \frac{\sum_{i=1}^{N} \alpha_i I_T^{\delta_i}}{\mathrm{E}\left(\sum_{i=1}^{N} \alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)}, \quad 0 \le t \le T, \tag{1}$$

with $\alpha_i, \delta_i \in \mathbb{R}$, $N \in \mathbb{N} \cup \{\infty\}$. To generate arbitrage-free asset prices the only restriction which has to be imposed on the parameters is that $0 < \phi_{t,T} < \infty$, P-a.s. This specification is rather general so that many different characteristics of the ASPK can be matched. Obviously the power function is a special case with $N = 1$ in equation (1). Since the ASPK based on the exponential function can be rewritten as

$$\phi_{t,T}^{exponential} = \frac{\sum_{k=0}^{\infty} \frac{1}{k!} (-a I_T)^k}{\mathrm{E}\left(\sum_{k=0}^{\infty} \frac{1}{k!} (-a I_T)^k \middle| \mathcal{F}_t\right)}, \quad 0 \le t \le T,$$

it follows that

$$\sum_{i=1}^{\infty} \widehat{\alpha}_i \exp(\widehat{\delta}_i I_T) = \sum_{i=1}^{\infty} \widehat{\alpha}_i \sum_{k=0}^{\infty} \frac{1}{k!} (\widehat{\delta}_i I_T)^k = \sum_{k=0}^{\infty} \left[\sum_{i=1}^{\infty} \frac{\widehat{\alpha}_i}{k!} (\widehat{\delta}_i)^k\right] (I_T)^k$$

under suitable conditions on $\widehat{\delta}_i, \widehat{\alpha}_i$. This proves that a sum of exponential functions is also a special case of the proposed polynomial. Furthermore, Lüders and Franke (2004) show that the generalized polynomial characterization approximates any ASPK at least as well as a Taylor expansion of the same order. This follows since a Taylor-series approximation of a function $f(x)$ about x_0 can be written as

$$\sum_{i=0}^{N} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i = \sum_{i=0}^{N} \frac{f^{(i)}(x_0)}{i!} \left(\sum_{k=0}^{i} \binom{i}{k} x^{i-k} (-x_0)^k\right) \tag{2}$$

where $f^{(i)}$ is the i^{th} derivative of f. Hence a Taylor-series approximation of order N is a special case of the generalized polynomial of order N.[7]

As will be shown later the flexibility of the ASPK is of great importance for option pricing but also for empirical investigations of option markets since recent empirical literature points to very complicated functional forms of empirical ASPKs. The main advantage of this new class of ASPKs, besides the fact that very flexible shapes of the asset specific pricing kernel can be well approximated, is that these ASPKs are characterized by a series of non-central moments of the random variable. Hence, for many distributions of the underlying asset, the ASPK and asset prices are easily computed. For example, Lüders and Franke (2004) show that the price F_t at time t of a lognormally distributed cash-flow I_T at time T is given by

$$F_t = \mathrm{E}(I_T \phi_{t,T} | \mathcal{F}_t), \quad 0 \le t \le T,$$

which can be rewritten as

$$F_t = \mathrm{E}\left(I_T \frac{\sum_{i=1}^{N} \alpha_i I_T^{\delta_i}}{\mathrm{E}(\sum_{i=1}^{N} \alpha_i I_T^{\delta_i} | \mathcal{F}_t)} \middle| \mathcal{F}_t \right), \quad 0 \le t \le T.$$

Thus, the price can be characterized by a sum of non-central moments.

In addition to these two more technical reasons for using a generalized polynomial ASPK there are also economic arguments which make the generalized polynomial ASPKs very interesting candidates for pricing options. The generalized polynomial ASPKs are straightforward to interpret and consistent with economic theory.

Therefore, consider first the case of a real option. N owners of a company have to decide whether or not to undertake a risky project. All owners have a similar payment schedule, they receive a proportion of the project's cash flow or final value, respectively. Hence, they all have a linear sharing rule with respect to this project. It is difficult to estimate risk preferences. Therefore it might be reasonable to assume that all owners of the company have utility functions with respect to the project under consideration that are characterized by a power function, i.e. CRRA or alternatively an exponential function, i.e. CARA. In this case only one parameter has to be estimated per owner. If, for example, every owner's project specific utility function is approximated by a power function, then the weighted objective function (the representative owner's utility function) would be $\sum_{i=1}^{N} U_i\,(\alpha_i I_T) = \sum_{i=1}^{N} (\alpha_i I_T)^{\delta_i}$. This leads to the proposed ASPK. For similar arguments we may use our approach also to price employee stock options which are often not traded.

Consider next the interesting characteristic of the polynomial ASPK that the elasticity is bounded from above and from below. This property of the ASPK follows also from models with heterogeneous agents (see for example Benninga and Mayshar, 2000). The analysis of the market portfolio in Benninga and Mayshar (2000) shows that in an economy with heterogeneous agents the representative agent's risk aversion is bounded from above and from below by the risk aversion of the most and the least risk averse investor. In a very simple economy the pricing kernel and the ASPK are the same and therefore the representative investor's risk aversion equals the elasticity of the ASPK.

In the following subsection we further elaborate the technical characteristics of the polynomial specification and we compare them to alternative approaches. These characteristics are especially important when it comes to applying our approach.

1.2. Technical properties of the pricing kernel class

The polynomial specification is a very flexible characterization which allows to approximate very general shapes of the true ASPK. As already mentioned, the specification can be interpreted as a generalized Taylor expansion. Therefore, very general forms of the ASPK can be approximated and the accuracy of this approximation is depending on the smoothness of the ASPK. Mathematically, the approximation error using $n-1$ terms is bounded by $\max|\phi_{t,T}^{(n)}|/n!$, where $\phi_{t,T}^{(n)}$ is the n-th derivative of the ASPK. Hence, one obtains a useful approximation for sufficiently large n, if the derivatives stay bounded.

Different parametric characterizations of the ASPK can be found in the literature. To back out empirical ASPKs from option prices Bartunek and Chowdhury (1997) assume a power utility function and an equity process with constant mean and volatility. This choice is very restrictive. Bliss and Panigirtzoglou (2004) also assume a restrictive form of the representative investor's utility function, either power or exponential utility, which both are special cases of specification (1). To infer risk-neutral probability density functions they use a smoothed weighted natural spline least-squares approximation of implied volatilities. This approach allows for non-stationary subjective probability density functions, but due to the restrictive form of the utility function their approach excludes by definition anomalies of the ASPK's form, e.g. non-monotonicity as observed by Jackwerth (2000).

Bliss and Panigirtzoglou (2004) find empirically that risk aversion implied by option data declines with the forecast horizon which implies that the ASPK is time-dependent. Note that characterization (1) can also be further generalized to allow for more flexibility in time by allowing the coefficients to be functions of time rather than being constant, without affecting the main results of this paper. We just note this possibility and do not pursue this any further here but leave it for future work.

Rosenberg and Engle (2002) propose two specifications of the ASPK. The first is a power function as in Bartunek and Chowdhury (1997) and Bliss and Panigirtzoglou (2004). The results in Rosenberg and Engle (2002) show that the specification as a simple power function restricts the form of the ASPK significantly.

In contrast, the more general specification (1) is more flexible and allows to approximate very general shapes of the true ASPK. As an example, we consider the ASPKs given in Jackwerth (2000)[8]. These are non-parametric estimates of ASPKs implied by S&P 500 options. From (1) we can compute the related elasticity,

$$\eta = -\frac{\partial \phi_{t,T}}{\partial I_T}\frac{I_T}{\phi_{t,T}} = -\frac{\sum \alpha_i \delta_i I_T^{\delta_i - 1} I_T}{\sum \alpha_i I_T^{\delta_i}}. \tag{3}$$

In order to fit the elasticity (3) of our specification to the empirical elasticities of Jackwerth (2000), we need to determine the coefficients α_i, δ_i, such that $\eta \approx \eta_{\text{emp}}$, with M given empirical elasticities $\eta_{\text{emp}} = (\eta_{\text{emp},k})_k$, $(k = 1, \dots, M)$, for different wealth. We use a nonlinear least squares approach, in particular a sub-space trust-region method (Coleman and Li, 1996), to minimize

$$\min_{\alpha_i,\delta_i} \sum_k (\eta_k - \eta_{\text{emp},k})^2.$$

Figure 1 shows a typical result. In this computation, we used specification (1) with $N = 5$, i.e. a sum of five terms. The resulting ASPK exhibits a non-monotonic behavior.[9]

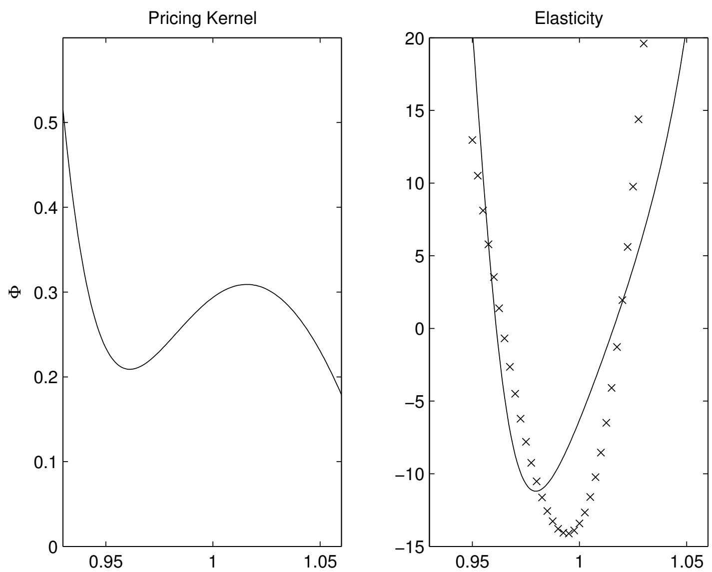

Figure 1. Fitting of empirical ASPKs. The right graph shows empirical elasticities of an ASPK implied by S&P 500 options (marked by crosses) and the fitted five-term polynomial specification (solid line), i.e. specification (4) with N=5. The ASPKs are published in Jackwerth (2000). The elasticity of the ASPK is u-shaped and reaches negative values as is typical for APSKs implied by S&P 500 options after the 1987 crash. The left graph is the ASPK that corresponds to the fitted elasticities. This ASPK is hump-shaped. While the left part of the elasticities is well fitted the fit for the right part is less satisfactory. However, the figure illustrates that even with only five terms the polynomial approximation can fit non-monotonic ASPKs reasonably well. The approximation error can be reduced by increasing the number of terms in the polynomial approximation.

The second specification proposed in Rosenberg and Engle (2002) is a weighted sum of orthogonal Chebyshev polynomials. This kind of orthogonal expansion has the advantage that it provides a comparably precise approximation with a low number of terms. The elements of specification (1) have no orthogonal property and hence may involve more terms. However, specification (1) has an advantage, which becomes very important if the goal is to obtain explicit formulas: It uses only powers of I_T. This allows for the derivation of explicit option pricing formulas, since only non-central moments of I_T have to be computed. For this purpose, explicit formulas exist for many approved underlying distributions.

A different approach is conducted by Jackwerth (2000), Ait-Sahalia and Lo (2000), and others. They use non-parametric approaches to infer the ASPK from empirical data. Jackwerth (2000) assumes a time-constant subjective probability density function (PDF) and compares time-series of subjective PDF to time-series of risk-neutral PDFs derived from S&P 500 data to obtain the ASPK. Non-parametric approaches do not restrict the form of the ASPK and can provide interesting results on the ASPKs behavior.

A parametric estimate of the true ASPK, however, has the advantage that it is possible to enforce certain characteristics of the ASPK by restricting the parameter set. For example, to ensure monotonicity and positivity of (1) we have the following sufficient conditions

- Monotonicity: $\delta_i \leq 0$,
- Positivity: $\alpha_i > 0$.

Further restrictions on the ASPK are easily imposed. To illustrate this we discuss now the restrictions suggested by Snow (1991). Snow (1991) proposes to study the q^{th} moments of the ASPK

$$\|\phi_{t,T}\|_q := \mathrm{E}[\phi_{t,T}^q]^{1/q}, \quad 1 < q < \infty.$$

As an example, we consider in the following the important case $q = 2$ for a special representative of (1), the two-term ASPK defined by

$$\phi_{t,T} = \frac{\frac{1}{I_T} + \beta I_T^\delta}{\mathrm{E}[\frac{1}{I_T} + \beta I_T^\delta | \mathcal{F}_t]}, \quad 0 \leq t \leq T, \tag{4}$$

with $\beta \geq 0$, $\delta \leq -1$. Note that using this special characterization of the ASPK yields an ASPK which is very close to the standard one with constant elasticity. However, this ASPK has declining elasticity. Figure 2 shows a contour plot of $\|\phi_{t,T}\|_2$ as a function of β and δ. Using this information it is simple to restrict the parameter set used for specification (1) in a way to meet with certain bounds for $\|\phi_{t,T}\|_2$. For example, if we use ASPK (4) to fit empirical data and restrict the parameter set for β, δ to the area above the dash-dotted line in Figure 2, we obtain an ASPK that fulfills the a-priori bound $\|\phi_{t,T}\|_2 < 1.2$.

This is particularly useful for approximation at the boundaries where no or only few data are available, and interpolation or extrapolation of these data, for example by splines, becomes problematic. Rosenberg and Engle (2002) perform their approximation in a moneyness region of $\pm 10\%$ and set the ASPK outside of this domain constant to its value at -10% and 10%, respectively. This seems unsatisfactory when compared to the monotonic behavior of ASPKs that stem from classic theory. In contrast, specification (1) allows for a monotonic behavior at the boundary (cf. Figure 1) and with restrictions imposed on the parameters as mentioned above it as well ensures certain properties of the ASPK. Thus, specification (1) provides a consistent approach to approximate the true ASPK also at the boarders, where only few data are available. Therefore, the polynomial ASPK characterization appears to be an appropriate approach also to alleviate the problem addressed, for example, in

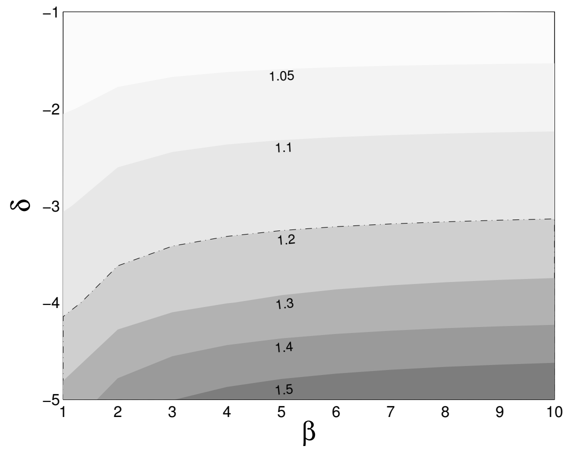

Figure 2. Contour plot of $||\Phi_{t,T}||_2$. The figure shows the contour plot of $||\Phi_{t,T}||_2$ for the two-term ASPK (4). It illustrates the combinations of β and δ that satisfy certain bounds on $||\Phi_{t,T}||_2$. For example, parameter combinations above the dash dotted line satisfy $||\Phi_{t,T}||_2 < 1.2$. With such restrictions it is possible to ensure that empirical estimates of ASPKs are not too erratic.

Constantinides, Jackwerth and Perrakis (2004), that reasonable ASPKs should not be too erratic.

Finally, the numeric effort is rather small, because of the little number of parameters that need to be determined. For example, in the case of the empirical Jackwerth (2000) ASPK, a sum of five terms suffices for a useful approximation and thus only ten parameters need to be determined. A cubic spline interpolation of the same data set involves more than 30 parameters.

Since our method is flexible in terms of the distribution as well as in terms of the ASPK it provides a new tool to infer ASPK from option data. To do this one would estimate the underlying distribution from past returns and then fit the option pricing formula to the observable option data. This would result in an analytical estimate of the ASPK.

In summary, although non-parametric approaches assure a great freedom for the form of the ASPK and new insight into the interplay of subjective and risk-neutral probability densities and the ASPK, they do not appear to be a suitable tool to derive option pricing formulas. Among the parametric approaches the orthogonal polynomial approach of Rosenberg and Engle (2002) as well as specification (1) are flexible enough to approximate general shapes of the true ASPK and are able to capture phenomena like non-monotonicity (increasing ASPK). In the context of option pricing, specification (1) seems commendable, since it admits the derivation of explicit pricing formulas.

2. Option Pricing

2.1. The general case

Our pricing methodology works in general as follows. In an arbitrage–free market the value of a European call (with expiration date T) at time $t \le T$ is given by

$$C_t = \mathrm{E}\left(\max\left(I_T - K, 0\right)\, \phi_{t,T} \,\middle|\, \mathcal{F}_t\right).$$

Assume that the asset specific pricing kernel is characterized by equation (1) and define $\mu(t, \delta_i) = \mathrm{E}\left(I_T^{\delta_i} \middle| \mathcal{F}_t\right)$. This yields

$$\begin{aligned} C_t &= \mathrm{E}\left(\max\left(\frac{\sum_{i=1}^N \alpha_i I_T^{\delta_i+1}}{\sum_{i=1}^N \alpha_i \mu(t,\delta_i)} - K \frac{\sum_{i=1}^N \alpha_i I_T^{\delta_i}}{\sum_{i=1}^N \alpha_i \mu(t,\delta_i)}, 0\right)\middle|\mathcal{F}_t\right) \qquad (5)\\ &= \int_K \frac{\sum_{i=1}^N \alpha_i I_T^{\delta_i+1}}{\sum_{i=1}^N \alpha_i \mu(t,\delta_i)} f(I_T,t)\, dI_T - K \int_K \frac{\sum_{i=1}^N \alpha_i I_T^{\delta_i}}{\sum_{i=1}^N \alpha_i \mu(t,\delta_i)} f(I_T,t)\, dI_T \\ &= \frac{\sum_{i=1}^N \left(\alpha_i \int_K I_T^{\delta_i+1} f(I_T,t)\, dI_T\right)}{\sum_{i=1}^N \alpha_i \mu(t,\delta_i)} - K \frac{\sum_{i=1}^N \left(\alpha_i \int_K I_T^{\delta_i} f(I_T,t)\, dI_T\right)}{\sum_{i=1}^N \alpha_i \mu(t,\delta_i)}, \end{aligned}$$

where $f(I_T, t)$ is the conditional density function of I_T. For the underlying asset this equation further simplifies to

$$F_t = \mathrm{E}\left(\frac{\sum_{i=1}^N \alpha_i I_T^{\delta_i+1}}{\sum_{i=1}^N \alpha_i \mu(t,\delta_i)}\middle|\mathcal{F}_t\right) = \frac{\sum_{i=1}^N \alpha_i \mu(t,\delta_i+1)}{\sum_{i=1}^N \alpha_i \mu(t,\delta_i)}. \qquad (6)$$

Equations (5) and (6) show that calculating prices under the assumption that the ASPK is characterized by a generalized polynomial basically reduces the pricing problem to calculating non-centralized moments. As we demonstrate by the following examples, for many distributions analytical solutions exist to these integrals and therefore analytical option prices can be calculated.

2.2. Lognormality

We consider a market with a given time horizon $T > 0$ and a one-dimensional standard Brownian motion W on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by W augmented by all the $\mathcal{F}$-null sets, with $\mathcal{F} = \mathcal{F}_T$. As in Franke, Stapleton and Subrahmanyam (1999) we assume that the information process – which can be interpreted as the representative investor's conditional expectation about the terminal value of the stock – is governed by a standard geometric Brownian motion without drift. Thus, we assume that the value of the underlying asset at expiration is given by I_T which is characterized by

$$\begin{aligned} dI_t &= \sigma I_t\, dW_t, \quad 0 \le t \le T, \\ I_0 &> 0, \end{aligned}$$

with constant σ. Hence, in this special case the terminal value I_T is lognormally distributed, as in the Black-Scholes model, with

$$\mathrm{E}(I_T|\mathcal{F}_t) = I_t \quad \text{and} \quad \mathrm{Var}(\ln I_T|\mathcal{F}_t) = \sigma^2(T-t), \quad 0 \le t \le T.$$

2.2.1. An analytical formula

In this section we consider the price of a European option with strike price K that expires at time T. The only assumption on the underlying asset is that its value at time T is lognormally distributed. From equation (5) and the fact that I_T is lognormally distributed it follows by rearranging terms that option prices in this ASPK class can be understood as a weighted sum of Black-Scholes prices,

$$C_t = \sum_{i=1}^{N} \underbrace{\frac{\mathrm{E}\left(\alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)}{\mathrm{E}\left(\sum_{i=1}^{N} \alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)}}_{\text{weightening factor}} \quad \underbrace{\frac{\mathrm{E}\left(\max\left(I_T - K, 0\right) \alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)}{\mathrm{E}\left(\alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)}}_{\text{Black-Scholes price for the virtual asset price } F_t^{(i)}} \tag{7}$$

$$= \sum_{i=1}^{N} \frac{\mathrm{E}\left(\alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)}{\mathrm{E}\left(\sum_{i=1}^{N} \alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)} \mathrm{BS}(t, F_t^{(i)}, K, \sigma), \quad 0 \le t \le T.$$

where

$$\mathrm{BS}(t, F_t^{(i)}, K, \sigma) = F_t^{(i)} N(d_1) - K N(d_2),$$

$$d_1 = \frac{\ln \frac{F_t^{(i)}}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

is the Black-Scholes formula for an asset price $F_t^{(i)}$. We call this a virtual asset price since $F_t^{(i)}$ is the price that would hold if the elasticity of the ASPK were δ_i. This virtual asset price $F_t^{(i)}$ is given by

$$F_t^{(i)} = \frac{\mathrm{E}\left(I_T \alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)}{\mathrm{E}\left(\alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t\right)} = \frac{\mathrm{E}\left(I_T^{\delta_i+1} \middle| \mathcal{F}_t\right)}{\mathrm{E}\left(I_T^{\delta_i} \middle| \mathcal{F}_t\right)}, \quad 0 \le t \le T,$$

and since I_T is lognormally distributed the conditional expectation of $I_T^{\delta_i}$ is given by

$$\mathrm{E}(I_T^{\delta_i}|\mathcal{F}_t) = \exp\left[\tfrac{1}{2}\delta_i^2\sigma^2(T-t) + \delta_i \mathrm{E}(\ln I_T|\mathcal{F}_t)\right].$$

This yields

$$F_t^{(i)} = I_t \exp\left[\delta_i \sigma^2(T-t)\right], \quad 0 \le t \le T.$$

To get a better understanding of equation (7) recall that in the Black-Scholes case the ASPK is given by a power function. If $N = 1$ then the first term in equation (7) is 1 and the option price is given by the classical Black-Scholes equation. If $N > 1$ then the option price is the weighted sum of Black-Scholes prices, where every Black-Scholes price $\mathrm{BS}(t, F_t^{(i)}, K, \sigma)$ corresponds to an economy with constant elasticity δ_i. The price of the underlying asset under the generalized ASPK is given by the weighted sum

$$F_t = \sum_{i=1}^{N} \underbrace{\frac{\mathrm{E}\left(\alpha_i I_T^{\delta_i} \,\middle|\, \mathcal{F}_t\right)}{\mathrm{E}\left(\sum_{i=1}^{N} \alpha_i I_T^{\delta_i} \,\middle|\, \mathcal{F}_t\right)}}_{\text{weightening factor}} F_t^{(i)}, \quad 0 \leq t \leq T. \tag{8}$$

The proposed class of ASPKs and a lognormally distributed terminal value therefore yield an analytical solution for European options which is given by a weighted sum of Black-Scholes prices. Note that this option pricing formula is written in terms of the expected value I_t of the underlying. This can be sometimes more convenient, when no market price for the underlying is available but the investor has some knowledge about the expected future payoff of the underlying. Moreover, an option pricing formula which depends explicitly on investors' expectations and on the parameters of the ASPK is a valuable tool to study the impact of expectations and preferences on option prices. In the case that asset prices are available, the observable price of the underlying is given by (8).

2.2.2. A generalized Black-Scholes partial differential equation

Let us now consider a European option that expires at time τ with $\tau < T$. Again, the only assumption on the underlying asset is that its value at time T is lognormally distributed. In order to analyze the relationship between the underlying asset price and the option price as well as the impact on the implied volatility, it is convenient to consider the partial differential equation which characterizes the option price, since the analytical formulas of section 2.2.1 are not applicable. We have

$$C_t = \mathrm{E}^Q \left(\max\left(F_\tau - K, 0\right) \middle| \mathcal{F}_t\right), \quad 0 \leq t \leq \tau,$$

where E^Q is the expected value with respect to the equivalent martingale measure Q. Further, in the Gaussian framework with continuous information diffusion the option price is a deterministic function $C = C(F, t)$ of the asset price F and time t. Hence, the option price is characterized by the following partial differential equation[10]

$$\frac{\partial C(F,t)}{\partial t} + \frac{1}{2}(\Sigma(F,t))^2 F^2 \frac{\partial^2 C(F,t)}{\partial F^2} = 0, \quad F > 0,\ 0 \leq t \leq \tau, \tag{9}$$

where $\Sigma(F, t)$ is the asset price process' volatility, with the final condition

$$C(F, \tau) = \max\left(F - K, 0\right), \quad F > 0. \tag{10}$$

Inserting our new class of ASPKs yields

$$\Sigma(F,t) = \frac{\sigma I \frac{\partial}{\partial I} F(I,t)}{F}$$

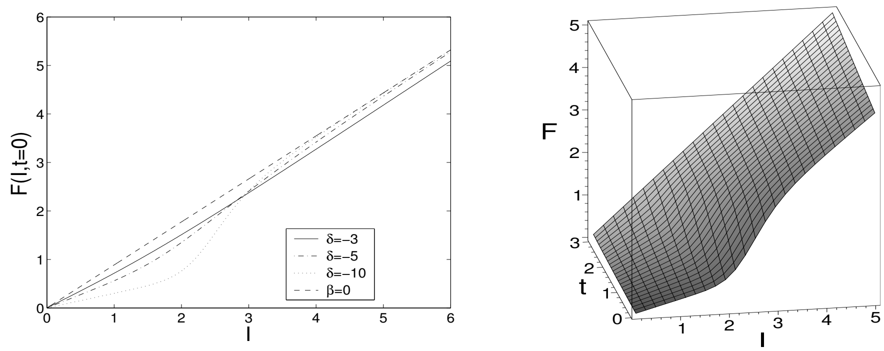

Figure 3. Forward prices under different ASPKs. The figure shows the forward price F as a function of the information process I (left graph) and the forward price F as a function of the information process I and time t (right graph). For both graphs we set terminal date $T = 3$ and the instantaneous volatility of the information process $\sigma = 0.2$. For the left graph we set $\beta = 5$ with $\delta = -3, -5, or - 10$. We also include the case with $\beta = 0$, i.e. constant elasticity of the ASPK. For the right graph $\delta = -10$, $\beta = 5$ and time t varies between 0 and 3.

with

$$F(I,t) = I \frac{\sum_{i=1}^{N} \alpha_i I^{\delta_i} A_i(t) \exp\left(\sigma^2 (T-t)\delta_i\right)}{\sum_{i=1}^{N} \alpha_i I^{\delta_i} A_i(t)},$$

with $A_i(t) = \exp(\frac{1}{2}\delta_i(\delta_i - 1)\sigma^2(T-t))$. A simple computation shows that (9) reduces to the standard Black-Scholes equation, in the case of the geometric Brownian motion with $F(I,t) = I \exp(\delta_i \sigma^2 (T-t))$.

To illustrate the influence of the ASPK on the price of a European call option we consider now (4) as a simple example of our general characterization of the ASPK. Again, let

$$\phi_{t,T} = \frac{\frac{1}{I_T} + \beta I_T^{\delta}}{\mathrm{E}[\frac{1}{I_T} + \beta I_T^{\delta} | \mathcal{F}_t]}, \quad 0 \le t \le T,$$

with $\beta \ge 0, \delta \le -1$. The forward price $F = F(I,t)$ is given by

$$F(I,t) = I \exp(\sigma^2(t-T)) \frac{1 + \beta I^{\delta+1} \exp((\delta^2 + \delta)\sigma^2(T-t)/2)}{1 + \beta I^{\delta+1} \exp((\delta^2 - \delta - 2)\sigma^2(T-t)/2)}. \qquad (11)$$

For $\beta = 0$ we recover the case of a geometric Brownian motion with $F(I,t) = I \exp(\sigma^2(t-T))$. The same holds for $\delta = -1$. Both $\beta = 0$ and $\delta = -1$ imply constant elasticity of the ASPK. Hence, for those cases we are in the classical Black–Scholes framework.

Figure 3 shows $F(I, t = 0)$ for $T = 3,\ \beta = 5,\ \delta = -3, -5, -10$ compared to the ASPK with constant elasticity ($\beta = 0$) as well as $F(I,t)$ for $T = 3,\ \beta = 5,\ \delta = -10$. The major deviation between ASPK (4) and the one with constant elasticity is for low levels of I and for times t far from maturity. Clearly, as $t \to T$, $F(I,t) \to I$, which coincides with the fact that $F_T = I_T$.

Equation (9) involves three variables I, F and t, since $\Sigma(F,t)$ depends explicitly on I. To remove I from the equation we would need to resolve equation (11) for I. In general, it is not clear how to achieve this because of the complex structure of (11). Therefore it is convenient to rewrite (9) in terms of the option price $\mathcal{C} = \mathcal{C}(I, t)$ as a function of I and t. To perform this transformation we need the following Lemma.

LEMMA 1. *Let $\sigma \geq 0$, $\beta > 0$ and $\delta \leq -1$. Then the forward price $F : (0, \infty) \times (0, \infty) \to \mathbb{R}$, $(I, t) \mapsto F(I, t)$, given by* (11)*, is strictly monotone in I and t.*

The proof is given in the Appendix.

Hence, the inverse $F^{-1}(\cdot, t)$, $(F, t) \mapsto F^{-1}(F, t) = I(F, t)$, exists and its derivative is $\frac{\partial I}{\partial F}(\cdot, t) = \left(\frac{\partial F}{\partial I}(\cdot, t)\right)^{-1}$, for each $0 \leq t \leq T$. Using the transformation

$$\mathcal{C}(I, t) = C(F, t)$$

we obtain (by the chain rule)

$$\frac{\partial C}{\partial F} = \frac{\partial \mathcal{C}}{\partial I}\frac{\partial I}{\partial F} = \frac{\partial \mathcal{C}}{\partial I}\left(\frac{\partial F}{\partial I}\right)^{-1}, \quad \frac{\partial^2 C}{\partial F^2} = \frac{\partial^2 \mathcal{C}}{\partial I^2}\left(\frac{\partial F}{\partial I}\right)^{-2} - \frac{\partial \mathcal{C}}{\partial I}\left(\frac{\partial F}{\partial I}\right)^{-2}\frac{\partial^2 F}{\partial I^2}.$$

With the transformation $\tilde{t} = \tau - t$ (and immediately dropping the tilde) (9), (10) becomes

$$\frac{\partial \mathcal{C}}{\partial t} - \frac{1}{2}\sigma^2 I^2 \frac{\partial^2 \mathcal{C}}{\partial I^2} + \frac{1}{2}\sigma^2 I^2 \frac{\partial^2 F}{\partial I^2}\frac{\partial \mathcal{C}}{\partial I} = 0, \quad I > 0,\ 0 \leq t \leq \tau \tag{12}$$

$$\mathcal{C}(I, 0) = \max(I - K, 0), \quad I > 0. \tag{13}$$

Note that (12) involves the second derivative of F with respect to I (where F is given by (11)). The complex structure of $\frac{\partial^2 F}{\partial I^2}$ (not given here) does not allow to find an explicit solution of (12), (13). Therefore we need to solve the problem numerically. We use a standard explicit finite difference scheme (forward Euler). For the computation we replace $\mathbb{R}^+$ by $[0, R]$ with $R > 0$. For simplicity, we consider a uniform grid $Z = \{I_i \in [0, R] : I_i = ih,\ i = 0, \ldots, N\}$ consisting of $N + 1$ grid points, with $R = Nh$ and with space step h and time step k. Let $\mathcal{C}_i^n$ denote the approximate solution of (12) in I_i at time $t_n = nk$ and set $\mathcal{C}^n = (\mathcal{C}_i^n)_{i=0}^N$. Dirichlet conditions are used on both boundary points:

$$\mathcal{C}_0^n = 0, \quad \mathcal{C}_N^n = Nh - K.$$

We choose the following parameters:

$$T = \tau = 0.1,\ t = 0,\ K = 1,\ \sigma = 0.2,\ N = 400,\ h = 0.01,\ R = 4.0,\ \beta = 5.$$

The choice of the underlying asset return's annual volatility of 20% is consistent with the 18% p.a. on the S&P 500. The solution of the original problem $C(F, t)$ is shown in Figure 4 for different values of δ. The option prices increase for smaller values of δ. Note that for $\delta = -1$, i.e. the ASPK has constant elasticity, the option price is lower than for declining elasticity. This is consistent with Theorem 1 in Franke, Stapleton and Subrahmanyam (1999) who show that option prices are ceteris paribus higher under declining elasticity of the ASPK than under constant elasticity of the ASPK.

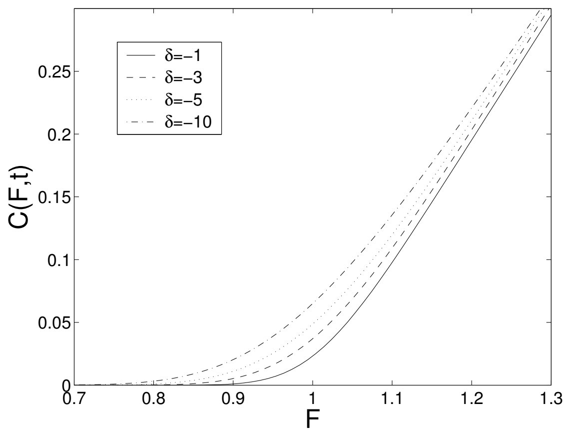

Figure 4. Option prices under different ASPKs. The figure shows the option price C as a function of the underlying forward price F for four different ASPKs. We use the two-term ASPK (4) with $\beta = 5$ and $\delta = -1, -3, -5, or - 10$. Time to maturity is $\tau = 0.1$, the instantaneous volatility is $\sigma = 0.2$, and the strike price is $K = 1$. The option prices are computed by numerical solution of PDE (12),(13). Note that the ASPK has constant elasticity for $\delta = -1$ and that the option prices are ceteris paribus higher for smaller values of δ.

2.3. The log-gamma case

The pricing of European options on a log-gamma distributed underlying is analogous to the pricing of options on lognormally distributed underlyings. While in the last chapter we assumed I_T to be lognormally distributed we assume now that I_T has a log-gamma distribution (Heston, 1993, p.937), i.e.

$$\ln I_T = \mu + \sigma z, \qquad g(z,\theta) = \begin{cases} \frac{z^{\theta-1}\exp(-z)}{\Gamma(\theta)}, & \text{for } 0 \leq z < \infty, \\ 0, & \text{for } z < 0, \end{cases}$$

where z has the gamma density $g(z,\theta)$ with degrees of freedom θ. The cumulative distribution function is given by

$$G(\bar{z};\theta) = \int_0^{\bar{z}} g(z,\theta)\, dz.$$

The log-gamma case is especially interesting for options on stocks and stock indices since the gamma distribution is flexible enough to capture the kurtosis and skewness of stock returns. Let us consider the corresponding information process. Since I_t is an information process, $I_t = \mathrm{E}\,(I_T | \mathcal{F}_t)$ and therefore $I_t = \exp(\mu)(1-\sigma)^{-\theta}$ with $\mu = \ln I_t + \theta \ln(1-\sigma)$. The degrees of freedom θ of the distribution depend on the time to maturity $(T-t)$. This can be easily seen if we consider the corresponding information process I_t which is given by

$$\ln I_t = \ln I_0 + \theta_x t \ln(1-\sigma) + \sigma x_t, \quad 0 \leq t \leq T,$$

where x_t is a slight generalization of the gamma process defined in Heston (1993, p.941). The process x_t has the property that $x_0 = 0$ a.s. and for $0 \leq s < t$, the increment $x_t - x_s$ is independent of $\mathcal{F}_s$ and has a gamma distribution with degrees of freedom $\theta_x(t-s)$. Hence the information process can also be written as

$$\ln I_t = \ln I_s + \theta_x(t-s)\ln(1-\sigma) + \sigma(x_t - x_s), \quad 0 \le s < t,$$

which implies that the degrees of freedom of the distribution of I_T conditional on the information $\mathcal{F}_t$ is given by $\theta_x(T-t)$. Heston (1993) argues that the degrees of freedom for monthly stock returns should be at least 6. Since we measure time in years, this implies $\theta_x = 72$.

Similarly to the derivation of option prices on lognormally distributed underlyings we can decompose our option pricing equation in terms which correspond to a world where the ASPK has constant elasticity δ_i. Heston (1993) derives an analytical option pricing formula for constant elasticity of the ASPK, hence the option price under our generalized ASPK is given by a weighted sum of Heston (1993) prices:

$$\begin{aligned}
C_t &= \mathrm{E}\left(\max(I_T - K, 0)\frac{\sum_{i=1}^N \alpha_i I_T^{\delta_i}}{\mathrm{E}\left(\sum_{i=1}^N \alpha_i I_T^{\delta_i}\middle|\mathcal{F}_t\right)}\middle|\mathcal{F}_t\right)\\
&= \mathrm{E}\left(\sum_{i=1}^N \frac{\max(I_T - K, 0)\alpha_i I_T^{\delta_i}}{\mathrm{E}\left(\sum_{i=1}^N \alpha_i I_T^{\delta_i}\middle|\mathcal{F}_t\right)}\middle|\mathcal{F}_t\right)\\
&= \sum_{i=1}^N \frac{\mathrm{E}\left(\alpha_i I_T^{\delta_i}\middle|\mathcal{F}_t\right)}{\mathrm{E}\left(\sum_{i=1}^N \alpha_i I_T^{\delta_i}\middle|\mathcal{F}_t\right)}\mathrm{E}\left(\frac{\max(I_T - K, 0)\alpha_i I_T^{\delta_i}}{\mathrm{E}\left(\alpha_i I_T^{\delta_i}\middle|\mathcal{F}_t\right)}\middle|\mathcal{F}_t\right) = \sum_{i=1}^N \varpi_i \ \mathrm{Heston}_i,
\end{aligned}$$

where

$$\mathrm{Heston}_i = F_t^{(i)}\left(1 - G(h_1;\theta)\right) - K\left(1 - G(h_2;\theta)\right),$$

for σ positive and

$$\mathrm{Heston}_i = F_t^{(i)} G(h_1;\theta) - K G(h_2;\theta),$$

for σ negative[11] with

$$h_1 = (\ln K - \mu)\frac{\left(\exp(-\mu)F_t^{(i)}\right)^{-1/\theta}}{1 - \left(\exp(-\mu)F_t^{(i)}\right)^{-1/\theta}}, \quad h_2 = h_1 + \ln K - \mu,$$

are the corresponding Heston (1993, p.939) option prices and ϖ_i are the weights.[12] $F_t^{(i)}$ is again the virtual asset price, defined as

$$F_t^{(i)} = \mathrm{E}\left(I_T \frac{I_T^{\delta_i}}{\mathrm{E}(I_T^{\delta_i}|\mathcal{F}_t)}\middle|\mathcal{F}_t\right) = e^{\mu}\frac{(1-(\delta_i+1)\sigma)^{-\theta}}{(1-\delta_i\sigma)^{-\theta}} = I_t \frac{(1-(\delta_i+1)\sigma)^{-\theta}}{(1-\sigma)^{-\theta}(1-\delta_i\sigma)^{-\theta}}.$$

Note that the generalized Heston option pricing equations have the same advantages as the generalized Black-Scholes pricing equations but they are more flexible with respect to the underlying distribution.

2.4. The case of a uniform distribution

If the underlying has a uniform distribution, then the density function of I_T is given by $f(I_T) = \frac{1}{b-a}$ for $I_T \in [a, b]$ and $f(I_T) = 0$ for $I_T \notin [a, b]$. Let us consider only the interesting case where $a < K < b$. Straightforward calculation shows that then the price of a European option for $0 \leq t \leq T$ is given by[13]

$$C_t = \mathrm{E}\left(\max(I_T - K, 0) \frac{\sum_{i=1}^N \alpha_i I_T^{\delta_i}}{\mathrm{E}\left(\sum_{i=1}^N \alpha_i I_T^{\delta_i} \,\middle|\, \mathcal{F}_t\right)} \,\middle|\, \mathcal{F}_t\right)$$

$$= \int_K^b \left((I_T - K) \frac{\sum_{i=1}^N \alpha_i I_T^{\delta_i}}{\mathrm{E}\left(\sum_{i=1}^N \alpha_i I_T^{\delta_i} \,\middle|\, \mathcal{F}_t\right)}\right) \frac{1}{b-a}\, dI_T$$

$$= \frac{\left[\sum_{i=1}^N \frac{\alpha_i}{\delta_i + 2} I_T^{\delta_i + 2}\right]_K^b - K \left[\sum_{i=1}^N \frac{\alpha_i}{\delta_i + 1} I_T^{\delta_i + 1}\right]_K^b}{\left[\sum_{i=1}^N \frac{\alpha_i}{\delta_i + 1} I_T^{\delta_i + 1}\right]_a^b},$$

when $\delta_i \notin \{-1, -2\}$; otherwise, similar equations hold in which $\ln(I_T)$ appears as the antiderivative of $1/I_T$. This provides us with a simple option pricing equation for rather general shapes of the ASPK and a uniformly distributed underlying.[14] A more widely used assumption is that the underlying is normally distributed. We analyze this case in the following section.

2.5. The case of a normal distribution

If the underlying is normally distributed and the ASPK is an exponential function, option prices are given by the Brennan (1979) formula. Brennan derives option prices under the assumption that the elasticity of the ASPK with respect to the underlying asset is proportional to the value of the underlying asset. Analogously to the previous sections we will price options by decomposing their value into option prices which would hold in the simple case. The following generalization of the Brennan (1979) formula is based on our previous result that a weighted sum of exponential functions is a special case of our generalized polynomial. Assuming that the ASPK can be characterized

$$\widehat{\phi}_{t,T} = \frac{\sum_{i=1}^N \widehat{\alpha}_i \exp(\widehat{\delta}_i I_T)}{\mathrm{E}\left(\sum_{i=1}^N \widehat{\alpha}_i \exp(\widehat{\delta}_i I_T) \,\middle|\, \mathcal{F}_t\right)}, \quad 0 \leq t \leq T,$$

and I_T is now normally distributed we get

$$C_t = \mathrm{E}\left(\max(I_T - K, 0)\widehat{\phi}_{t,T} \,\middle|\, \mathcal{F}_t\right)$$

$$= \sum_{i=1}^N \frac{\mathrm{E}\left(\widehat{\alpha}_i \exp(\widehat{\delta}_i I_T) \,\middle|\, \mathcal{F}_t\right)}{\mathrm{E}\left(\sum_{i=1}^N \widehat{\alpha}_i \exp(\widehat{\delta}_i I_T) \,\middle|\, \mathcal{F}_t\right)} \mathrm{E}\left(\frac{\widehat{\alpha}_i \exp(\widehat{\delta}_i I_T)}{\mathrm{E}\left(\widehat{\alpha}_i \exp(\widehat{\delta}_i I_T) \,\middle|\, \mathcal{F}_t\right)} \max(I_T - K, 0) \,\middle|\, \mathcal{F}_t\right)$$

$$= \sum_{i=1}^N \widehat{\kappa}_i \ \mathrm{Brennan}_i, \quad 0 \leq t \leq T,$$

where

$$\text{Brennan}_i = \left(F_t^{(i)} - K\right) N\left(\frac{F_t^{(i)} - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\, n\left(\frac{K - F_t^{(i)}}{\sigma\sqrt{T-t}}\right)$$

are the corresponding Brennan (1979) option prices and $\widehat{\kappa}_i$ are the weights. $N(\cdot)$ is again the cumulative standard normal density function and $n(\cdot)$ denotes the standard normal density function. $F_t^{(i)}$ is the virtual asset price with

$$F_t^{(i)} = \mathrm{E}\left(\frac{I_T\widehat{\alpha}_i \exp(\widehat{\delta}_i I_T)}{\mathrm{E}\left(\widehat{\alpha}_i \exp(\widehat{\delta}_i I_T)\middle|\mathcal{F}_t\right)}\middle|\mathcal{F}_t\right) = I_t + \widehat{\delta}_i\sigma^2(T-t), \quad 0 \le t \le T.$$

The information process would then be given by a Brownian motion with constant volatility σ and no drift, i.e.,

$$\begin{aligned} dI_t &= \sigma\, dW_t, \quad 0 \le t \le T, \\ I_0 &> 0. \end{aligned}$$

In this case, the terminal value I_T is normally distributed with

$$\mathrm{E}(I_T|\mathcal{F}_t) = I_t \quad \text{and} \quad \mathrm{Var}(I_T|\mathcal{F}_t) = \sigma^2(T-t), \quad 0 \le t \le T.$$

Hence, this ASPK specification generates analytical option pricing formulas for normally distributed underlyings although the ASPK is not an exponential function and hence the elasticity is not linear in the underlying asset.

Our previous derivations show that the option pricing approach proposed in this paper would be consistent with many alternative distributional assumptions. For example Cox, Ross and Rubinstein's (1979) binomial formula and Heston's (1993) option pricing formula, based on the negative binomial density, as well as several other option pricing formulas are consistent with a power function as ASPK, hence they can be easily extended to the case where the ASPK is given by (1).

2.6. Implied Volatilities

To compute the implied volatilities of the option prices we use the following iteration procedure. Let $\mathcal{C}$ be the option price computed by one of the formulas of section 2 and let $\sigma^{(0)}$ be a given starting value. Then,

- For a given volatility $\sigma^{(n)}$ compute the Black-Scholes option price $C(\sigma^{(n)})$,
- Compute $\sigma^{(n+1)} = \sigma^{(n)} - \frac{C(\sigma^{(n)})-\mathcal{C}}{C'(\sigma^{(n)})}$,
- Set $n := n + 1$, repeat cycle.

Let $\sigma_i^{(n)}$ denote the n^{th} iterate of the implied volatility at grid point I_i. We stop the iteration procedure when the l_2 norm of the update defined by

$$\varepsilon_2 = \left(h\sum_{i=0}^{N}\left|\sigma_i^{(n+1)} - \sigma_i^{(n)}\right|^2\right)^{\frac{1}{2}}$$

becomes less than 10^{-5}.

2.6.1. *Different distributions with maturity* $\tau = T$

Using the following parameters

$$T = 0.2,\ t = 0,\ \sigma = 0.2,\ \ \beta = 5,\ \delta = -10,$$

we compute the implied volatilities in the following settings

- ASPK (4) with lognormal distribution of I_T,
- ASPK (4) with log-gamma distribution of I_T,
- ASPK (4) with normal distribution of I_T,
- Empirical ASPK (Figure 1) with lognormal distribution of I_T.

For the log-gamma case we set the additional parameter $\theta_x = 72$. The results are shown in Figure 5. For I_T lognormally distributed and an ASPK with declining elasticity, ASPK (4), we observe (see top left of Figure 5) a significant volatility skew. That is the implied volatility for in-the-money calls (i.e. out-of-the-money puts) is significantly higher than the implied volatility of at-the-money calls and out-of-the-money calls. This effect is similar to the case where we use a lognormal distribution and the fitted empirical ASPK, however, with the empirical ASPK the volatility skew is more pronounced (see bottom right of Figure 5). For a normally distributed underlying and ASPK (4) we find an inversed volatility skew, that is out-of-the-money calls have the highest implied volatility (see bottom left of Figure 5). The same holds for the log-gamma case (see top right of Figure 5). Why is the implied volatility skew inversed in these two cases? This is easily illustrated for the log-gamma distribution. Note that for Figure 5 and 6 we used a positive sigma and this implies for the log-gamma distribution that asset returns are positively skewed. Positive skewness of returns enhances the probability of extreme positive returns which consequently increases the value of out-of-the-money calls compared to the Black-Scholes case. While declining elasticity of the ASPK compared to constant elasticity of the ASPK increases the value of in-the-money calls (i.e. out-of-the-money puts) positive skewness tends to increase the value of out-of-the-money calls (i.e. in-the-money puts).

Figure 6 illustrates this effect of higher moments on option prices. There we consider the case that I_T is log-gamma distributed with different values of θ_x. We set $T = 0.1$, $t = 0$, $\sigma = 0.2$ and compute the implied volatilities related to two different ASPKs, the ASPK with constant elasticity $\delta = -1$ and ASPK (4) with $\beta = 5$, $\delta = -10$. The implied volatilities are shown in Figure 6. The log-gamma prices are computed using the formula in section 2.3. Therein, the parameter μ is chosen in a way as to approximate the Black-Scholes pricing of at-the-money options, namely $\mu = \ln(F_t^{(i)}) - \sigma\sqrt{\theta}$ with degrees of freedom $\theta = \theta_x(T - t)$.

Hence, the implied volatility for the ASPK with constant elasticity ($\delta = -1$) is approximately equal to $\sigma = 0.2$ at-the-money. The implied volatility has a negative slope. This is consistent with the fact, that Heston's formula (for positive σ) assigns

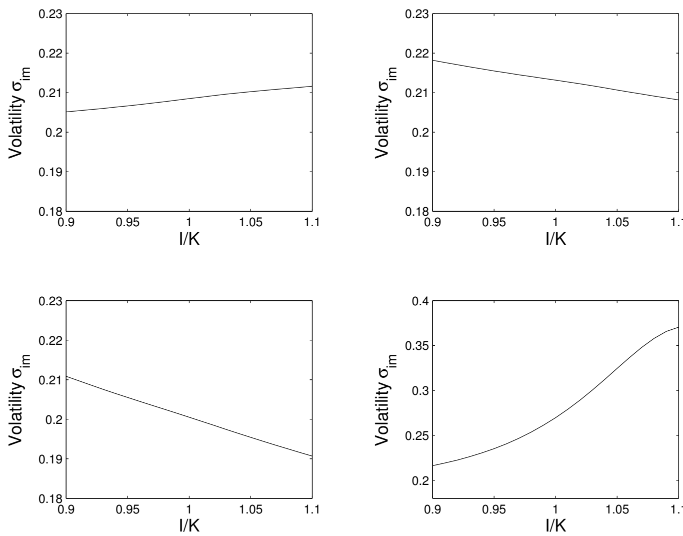

Figure 5. Implied volatilities under different ASPKs and different distributions. The figure shows the implied volatilities for four different settings. The graphs top left, top right and bottom left are all based on the two-term ASPK (4) with $\delta = -10$ and $\beta = 5$. For the graph bottom right we use the five-term ASPK fitted to the empirical ASPK. The underlying is lognormally distributed for the figures top left and bottom right. For the figure top right we use a log-gamma distribution with the additional parameter $\theta_x = 72$. For the figure on the bottom left we use a normal distribution. The time to maturity is $\tau = T = 0.2$ and $\sigma = 0.2$ for all four settings.

higher prices to out-of-the-money call options and lower prices to in-the-money call options, compared to the Black-Scholes formula. As θ_x grows large, the gamma distribution approaches the normal distribution and the implied volatility approaches the Black-Scholes value for all values of I/K.

In Figure 7 we plot the implied volatility for the log-gamma distribution and a negative sigma. Hence, in this case asset returns are negatively skewed which is consistent with empirical findings for stock prices, especially major stock indices. We see that with negatively skewed asset returns the implied volatility is higher for in-the money calls (i.e. out-of-the-money puts) than for at-the-money calls (i.e. at-the-money puts) and out-of-the money calls (i.e. in-the-money puts). Since negative skewness and declining elasticity of the ASPK work in the same direction we observe a steeper volatility skew for declining elasticity of the ASPK than for constant elasticity of the ASPK.

2.6.2. Lognormality for maturities $\tau < T$

For a lognormally distributed underlying we turn to the case of options expiring at times τ with $\tau < T$. Applying the same method as in the previous section we compute the implied volatilities of the option price given by the numerical solution of (12), (13) for different maturities τ. During the iteration procedure we need to compute the Black-Scholes price of the option with respect to the forward price

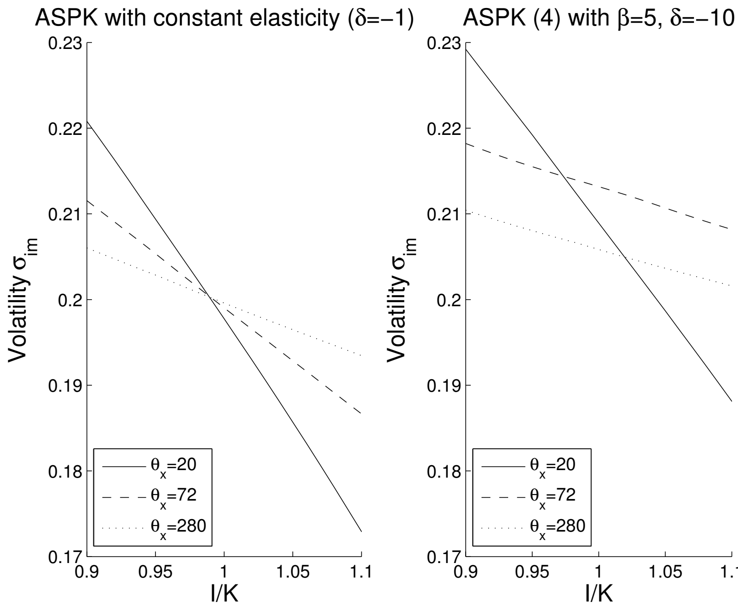

Figure 6. Implied volatilities under different ASPKs and log-gamma distributed underlyings with positive σ. The figure shows the implied volatilities for two different ASPKs. For the left graph, the ASPK has constant elasticity, i.e. ASPK (4) with $\delta = -1$ and $\beta = 5$. The ASPK in the right graph has declining elasticity, i.e. ASPK (4) with $\delta = -10$ and $\beta = 5$. For both graphs we use a log-gamma distributed underlying with $\theta_x = 20$ (solid line), $\theta_x = 72$ (dashed line) or $\theta_x = 280$ (dotted line). σ is positive and equal to 0.2 and time to maturity is $T = 0.1$.

F on an equidistant grid. Since the numerical solution of (12), (13) is a function of I, we use an interpolation with piecewise polynomials (cubic spline) to obtain Black-Scholes prices at the grid points of Z. We choose the following parameters:

$$T = 0.1,\ t = 0,\ \sigma = 0.2,\ N = 400,\ R = 4.0,\ \beta = 5,\ \delta = -10.$$

Figure 8 shows the implied volatility. The minimal value is $\sigma_{\text{im}}(0.99, 0.05) = 0.1469$. The implied volatility increases with growing maturity τ. Note also that the implied volatility is not symmetric. It is steeper in-the-money than out-of-the-money. These characteristics of the implied volatility are consistent with the empirically observed patterns of implied volatilities of S&P 500 options[15].

3. Conclusion

In this paper we derive analytical option pricing formulas for very flexible shapes of the ASPK and many different distributions of the underlying asset. These option pricing equations are based on a generalized polynomial characterization of the ASPK. Technically speaking the polynomial characterization has the main advantage of being very flexible and allowing for analytical option pricing formulas. Furthermore it

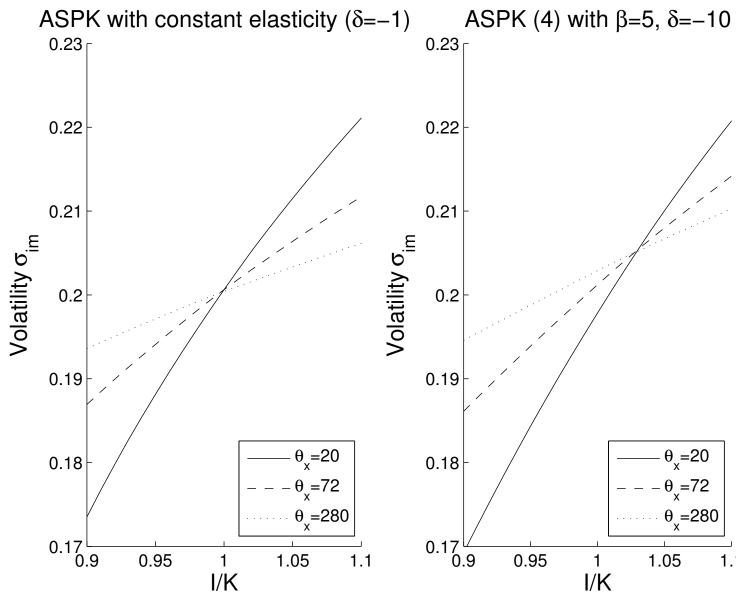

Figure 7. Implied volatilities under different ASPKs and log-gamma distributed underlyings with negative σ. The figure shows the implied volatilities for two different ASPKs. For the left graph, the ASPK has constant elasticity, i.e. ASPK (4) with $\delta = -1$ and $\beta = 5$. The ASPK in the right graph has declining elasticity, i.e. ASPK (4) with $\delta = -10$ and $\beta = 5$. For both graphs we use a log-gamma distributed underlying with $\theta_x = 20$ (solid line), $\theta_x = 72$ (dashed line) or $\theta_x = 280$ (dotted line). σ is negative and equal to - 0.2 and time to maturity is $T = 0.1$.

allows for reasonable approximations of empirical ASPKs with a small number of parameters.

Our approach to derive analytical option pricing formulas is widely applicable. First, in option pricing the derived analytical option pricing equations are more flexible than existing analytical option pricing formulas and should therefore prove to have an enhanced pricing accuracy compared to alternative analytical option pricing equations.

Since the option pricing formulas depend on the expected value of the underlying asset and no price of the underlying asset is needed, our approach is also an interesting model for options on non-traded underlyings where a market value of the underlying is not available.

Due to the flexibility of the polynomial characterization of the ASPK our approach will help to evaluate the quantitative implications of non-constant elasticity of the ASPK on option prices. Given our option pricing formulas it is straightforward to analyze the quantitative impact of alternative assumptions on the underlying asset and the ASPK on option prices. Up to now, only qualitative results on these relations were known, see e.g. Franke, Stapleton and Subrahmanyam (1999).

The numerical properties of the polynomial characterization of the ASPK point out that our new approach may also be used to infer empirical ASPK from option prices. It somewhat combines the advantages of existing parametric approaches

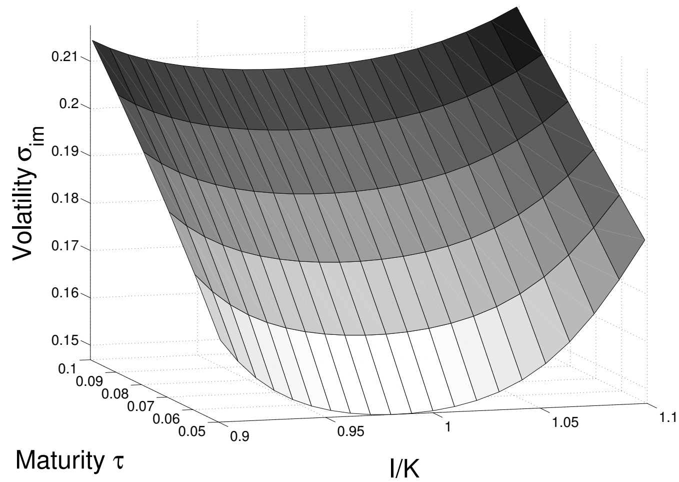

Figure 8. Implied volatilities for different maturities $\tau \leq T$. The figure shows the implied volatilities under the two-term ASPK (4) for different maturities $0.05 \leq \tau \leq 0.1$. The instantaneous volatility of the information process is $\sigma = 0.2$, current time is $t = 0$, and the terminal date $T = 0.1$. The ASPK has declining elasticity with $\beta = 5$ and $\delta = -10$. The option prices are computed by numerical solution of PDE (12), (13). Note that the implied volatility is skewed and increases with growing maturity τ.

with the greater flexibility of nonparametric approaches. It therefore seems to be a promising approach for future empirical analysis of ASPKs. We leave the empirical implementation for future research.

Appendix A. Proof of Lemma 1

We obtain from (11)

$$
\begin{aligned}
\frac{\partial F}{\partial I} =& \frac{e^{\sigma^2(t-T)}(1+\beta(\delta+2)I^{\delta+1}e^{\frac{1}{2}(\delta^2+\delta)\sigma^2(T-t)})}{1+\beta I^{\delta+1}e^{\frac{1}{2}(\delta^2-\delta-2)\sigma^2(T-t)}} \\
&- \frac{e^{\sigma^2(t-T)}(1+\beta I^{\delta+1}e^{\frac{1}{2}(\delta^2+\delta)\sigma^2(T-t)})\beta(\delta+1)I^{\delta+1}e^{(\frac{1}{2}(\delta^2-\delta-2)\sigma^2(T-t))}}{(1+\beta I^{(\delta+1)}e^{\frac{1}{2}(\delta^2-\delta-2)\sigma^2(T-t)})^2} \\
=& \Big[\big(1+\beta(\delta+2)I^{\delta+1}e^{\frac{1}{2}\sigma^2(\delta^2+\delta)(T-t)} + \beta^2 I^{2\delta+2}e^{\sigma^2(\delta^2-1)(T-t)} \\
&- \beta\delta I^{\delta+1}e^{\frac{1}{2}\sigma^2(\delta^2-\delta-2)(T-t)}\big)e^{\sigma^2(t-T)}\Big]\Big[1+\beta I^{\delta+1}e^{\frac{1}{2}(\delta^2-\delta-2)\sigma^2(T-t)}\Big]^{-2}.
\end{aligned}
$$

The denominator is positive, therefore we only have to consider the numerator,

$$
\begin{aligned}
N :=& 1 + 2\beta I^{\delta+1}e^{\frac{1}{2}\sigma^2(\delta^2+\delta)(T-t)} + \beta^2 I^{2\delta+2}e^{\sigma^2(\delta^2-1)(T-t)} \\
&+ \beta\delta I^{\delta+1}\big(e^{\frac{1}{2}\sigma^2(\delta^2+\delta)(T-t)} - e^{\frac{1}{2}\sigma^2(\delta^2-\delta-2)(T-t))}\big).
\end{aligned}
$$

The first three terms are non-negative or positive, respectively. Since $(\delta^2+\delta)-(\delta^2-\delta-2) = 2\delta+2 \leq 0$, we have $e^{-\frac{1}{2}\delta\sigma^2(\delta+1)(t-T)} - e^{-\frac{1}{2}\sigma^2(\delta+1)(\delta-2)(t-T)} \leq 0$ and the fourth term is non-negative, too. Furthermore

$$\frac{\partial F}{\partial t} = \frac{I\sigma^2 e^{\sigma^2(t-T)}[(1+\beta I^{\delta+1}e^{(\delta^2+\delta)\frac{\sigma^2}{2}(T-t)}) - (\frac{1}{2}\beta I^{\delta+1}(\delta^2+\delta)e^{(\delta^2+\delta)\frac{\sigma^2}{2}(T-t)})]}{1+\beta I^{\delta+1}e^{(\delta^2-\delta-2)\frac{\sigma^2}{2}(T-t)}}$$
$$+ \frac{I\sigma^2 e^{\sigma^2(t-T)}(1+\beta I^{\delta+1}e^{(\delta^2+\delta)\frac{\sigma^2}{2}(T-t)})\beta I^{\delta+1}(\delta^2-\delta-2)e^{(\delta^2-\delta-2)\frac{\sigma^2}{2}(T-t)}}{2(1+\beta I^{\delta+1}e^{(\delta^2-\delta-2)\frac{\sigma^2}{2}(T-t)})^2}.$$

A computation similar as above shows that $\frac{\partial F}{\partial t} > 0$.

Notes

1 The asset specific pricing kernel is the pricing kernel conditioned on the payoffs of an asset. For a detailed discussion of the pricing kernel and the asset specific pricing kernel we refer the reader to the excellent textbook of Cochrane (2001) as well as to the articles of Câmara (2001) and Câmara (2003).

2 This is, for example, analyzed in depth by Câmara (2003).

3 Closely related are also articles that analyze heterogeneous expectations and the consequences for option pricing, see for example Huang (2003) and Ziegler (2002).

4 Alternatively asset prices could be interpreted as forward prices.

5 Here and in the following E denotes the expected value with respect to the subjective measure P.

6 Assuming such an exogenous information process to model the information in the economy is common. The main advantage of this approach is that it is a parsimonious and intuitive way to characterize the filtration, see Franke, Stapleton and Subrahmanyam (1999).

7 See Lüders and Franke (2004).

8 The data on empirical elasticities of ASPKs were kindly provided by Jens Jackwerth. These empirical ASPKs are published in Jackwerth (2000).

9 The specific result also depends on the conditions that are imposed on the behavior for levels of moneyness, where no empirical data is available.

10 This follows from the Theorem of Feynman-Kac (see, e.g. Karatzas and Shreve, 1991). In order to apply the Theorem of Feynman-Kac the expected value has to exist. Since the underlying asset is basically a weighted sum of lognormally distributed assets this requirement holds.

11 This formula corrects a typing error in formula (10b) in Heston (1993).

12 It follows from the assumptions on the information process that the degrees of freedom θ are given by $\theta = \theta_x(T-t)$.

13 See also Haugen (2001) for a presentation of option prices for uniformly distributed underlyings under the assumption of risk-neutrality.

14 Note also that for the assumed uniform distribution for I_T, $\mathrm{E}(I_T^{\delta_i}|\mathcal{F}_t)$ can be rewritten as $\sum_{k=0}^{\delta_i}\binom{\delta_i}{k}\frac{a^k(a-b)^{\delta_i-k}}{1+\delta_i-k}$.

15 See Rubinstein (1994), Jackwerth and Rubinstein (1996), Ait-Sahalia and Lo (1998) and Carr and Wu (2003).

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