On the stability of a compact finite difference scheme for option pricing

Bertram Düring and Michel Fournié

Abstract In this short paper we are concerned with the von Neumann stability analysis of a compact high-order finite difference scheme for option pricing in the Heston stochastic volatility model. We first review results on the unconditional stability in the case of vanishing correlation and then present some new results on the behavior of the amplification factor for non-zero correlation.

1 Introduction

The Heston model [8] is a stochastic volatility model for option pricing where the option price V as function of price of the underlying S, volatility σ and time t solves

$$V_t + \tfrac{1}{2}S^2\sigma V_{SS} + \rho v\sigma S V_{S\sigma} + \tfrac{1}{2}v^2\sigma V_{\sigma\sigma} + rSV_S + \left[\kappa^*(\theta^* - \sigma) - \lambda\sigma\right]V_\sigma - rV = 0, \tag{1}$$

for $S,\sigma > 0$, $0 \le t < T$ and subject to a suitable final condition, e.g. $V(S,\sigma,T) = \max(K-S,0)$, in case of a European put option with strike price K. In (1), κ^*, v, θ^*, and λ denote the constant mean reversion speed, volatility of volatility, long-run mean of volatility, and market price of volatility risk parameter, respectively. The ‘boundary’ conditions in the case of the put option read as follows

$$V(0,\sigma,t) = Ke^{-r(T-t)}, \quad T > t \ge 0,\ \sigma > 0,$$
$$V(S,\sigma,t) \to 0, \quad T > t \ge 0,\ \sigma > 0,\ \text{as } S \to \infty,$$

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$$V_\sigma(S,\sigma,t) \to 0, \quad T > t \geq 0,\ S > 0,\ \text{as } \sigma \to 0 \text{ and } \sigma \to \infty.$$

For *constant* parameters, one can employ Fourier transform techniques and obtain a system of ordinary differential equations which can be solved analytically [8]. In general, however, when the coefficients are not constant, equation (1) has to be solved numerically. In the mathematical literature, there are many papers on numerical methods for option pricing with a single underlying. Most approaches use standard, second order finite difference methods. Compact high-order finite difference schemes were proposed, e.g. in [6, 7, 12]. For option pricing in the Heston model different second order finite difference methods for solving the American option pricing problem are compared in [10]. In [9] different, low order ADI (alternating direction implicit) schemes are adapted to the Heston model to include the mixed spatial derivative term. Other approaches include finite element-finite volume [16], multigrid [2], sparse wavelet [11], and spectral methods [15].

In [3, 4] we proposed a new *high-order compact finite difference scheme* for option pricing in the Heston model. It can easily be adapted to other stochastic volatility models (e.g. [5]). In this short paper we focus on the von Neumann stability analysis of the new scheme. We first review stabilty results that we obtained in [3] in the case of vanishing correlation, i.e. $\rho = 0$. Then we present some new results on the behavior of the amplification factor for non-zero correlation.

2 High order compact scheme

Let us introduce the modified parameters, $\kappa = \kappa^* + \lambda$, $\theta = \kappa^*\theta^*/(\kappa^* + \lambda)$, which allow us to study the problem with one parameter less. Under the transformation of variables $x = \ln(S/K)$, $y = \sigma/v$, $\tilde{t} = T - t$, $u = \exp(r\tilde{t})V/K$, (we immediately drop the tilde in the following) and using the modified parameters, κ and θ, we then obtain from (1),

$$u_t - \tfrac{1}{2}vy(u_{xx} + u_{yy}) - \rho v y u_{xy} + \left(\tfrac{1}{2}vy - r\right)u_x - \kappa\tfrac{\theta - vy}{v}u_y = 0, \tag{2}$$

which has to be solved on $\mathbb{R} \times \mathbb{R}^+$ with transformed initial and boundary conditions.

For the discretization, we replace $\mathbb{R}$ by $[-R_1, R_1]$ and $\mathbb{R}^+$ by $[0, R_2]$ with $R_1, R_2 > 0$. For simplicity, we consider a uniform grid with mesh width h in both the x- and y-direction, $Z = \{x_i \in [-R_1, R_1] : x_i = ih,\ i = -N, \ldots, N\} \times \{y_j \in [0, R_2] : y_j = jh,\ i = 0, \ldots, M\}$ consisting of $(2N+1) \times (M+1)$ grid points, with $R_1 = Nh$, $R_2 = Mh$ and time step k. Let $u^n_{i,j}$ denote the approximate solution of (2) in (x_i, y_j) at the time $t^n = nk$ and let $u^n = (u^n_{i,j})$. On the truncated numerical domain we impose artificial boundary conditions. We difference at time $t_\mu = (1-\mu)t^n + \mu t^{n+1}$, where $0 \leq \mu \leq 1$. This yields a class of integrators that include the forward Euler ($\mu = 0$), Crank-Nicolson ($\mu = 1/2$) and backward Euler ($\mu = 1$) schemes.

The fourth-order compact finite difference scheme derived in [3] uses a nine-point computational stencil involving the eight nearest neighboring points of the reference grid point (i, j),

$$\begin{pmatrix} u_{i-1,j+1}=u_6 & u_{i,j+1}=u_2 & u_{i+1,j+1}=u_5 \\ u_{i-1,j}=u_3 & u_{i,j}=u_0 & u_{i+1,j}=u_1 \\ u_{i-1,j-1}=u_7 & u_{i,j-1}=u_4 & u_{i+1,j-1}=u_8 \end{pmatrix}.$$

The resulting fully discrete difference scheme for node (i, j) at the time level n can be written in the form

$$\sum_{l=0}^{8} \beta_l u_l^{n+1} = \sum_{l=0}^{8} \zeta_l u_l^n, \tag{3}$$

where the coefficients β_l, ζ_l are given by (a detailed derivation is presented in [3])

$$\begin{aligned}
\beta_0 =&(((2{y_j}^2-8)v^4+((-8\kappa-8r)y_j-8\rho r)v^3+(8\kappa^2{y_j}^2+8r^2)v^2\\
&-16\kappa^2\theta v y_j+8\kappa^2\theta^2)\mu k+16v^3y_j)h^2+(-16\rho^2+40){y_j}^2v^4\mu k,\\
\beta_{1,3} =&\pm((\kappa\theta v^2-v^4-\kappa y_j v^3)\mu k-(y_j+2\rho)v^3+2v^2r)h^3+(((-{y_j}^2+2)v^4\\
&+((4r+2\kappa)y_j+4\rho r)v^3-(2\kappa\theta+4r^2)v^2)\mu k+2v^3y_j)h^2\\
&\pm(4v^4{y_j}^2+(-8{y_j}^2\kappa\rho-8y_j r)v^3+8y_j\kappa\theta\rho v^2)\mu kh+(8\rho^2-8){y_j}^2v^4\mu k,\\
\beta_{2,4} =&\pm((2\kappa^2\theta v-2\kappa^2v^2y_j-2v^3\kappa)\mu k-2v^2y_j\kappa+2v\kappa\theta-2v^3)h^3+((2v^4\\
&+2\kappa y_j v^3+(-4\kappa^2{y_j}^2+2\kappa\theta)v^2+8\kappa^2\theta v y_j-4\kappa^2\theta^2)\mu k+2v^3y_j)h^2\\
&\pm((8{y_j}^2\kappa+8y_j\rho r)v^3-4v^4{y_j}^2\rho-8v^2y_j\kappa\theta)\mu kh+(8\rho^2-8){y_j}^2v^4\mu k,\\
\beta_{5,7} =&((v^4\rho+(-y^2\kappa+\kappa y_j\rho+r)v^3+(\theta+2r)\kappa y_j v^2-2r\kappa\theta v)\mu k\\
&+v^3\rho y_j)h^2\pm((2\rho+1){y_j}^2v^4+((2+4\rho)\kappa{y_j}^2+(-4\rho r-2r)y_j)v^3\\
&+(-2\theta-4\theta\rho)\kappa y_j v^2)\mu kh+(-2-4\rho^2-6\rho){y_j}^2v^4\mu k,\\
\beta_{6,8} =&((-v^4\rho+({y_j}^2\kappa-\kappa y_j\rho-r)v^3+(-\theta-2r)\kappa y_j v^2+2r\kappa\theta v)\mu k\\
&-v^3\rho y_j)h^2\pm((2\rho-1){y_j}^2v^4+((2-4\rho)\kappa{y_j}^2+(2r-4\rho r)y_j)v^3\\
&+(4\theta\rho-2\theta)\kappa y_j v^2)\mu kh+(-4\rho^2+6\rho-2){y_j}^2v^4\mu k,\\
\zeta_0 =&16v^3y_jh^2+(1-\mu)k(((8-2{y_j}^2)v^4+((8\kappa+8r)y_j+8\rho r)v^3\\
&+(-8r^2-8\kappa^2{y_j}^2)v^2+16\kappa^2\theta v y_j-8\kappa^2\theta^2)h^2+(-40+16\rho^2){y_j}^2v^4),\\
\zeta_{1,3} =&\pm(2r-(y_j+2\rho)v)v^2h^3+2v^3y_jh^2+(1-\mu)k(\pm(v\kappa y_j+v^2-\kappa\theta)v^2h^3\\
&+(v^2{y_j}^2-(4r+2\kappa)v y_j+4r^2+2\kappa\theta-2v^2-4\rho v r)v^2h^2\\
&\pm((-4v+8\kappa\rho)v^3{y_j}^2+(-8\kappa\theta\rho+8vr)v^2y_j)h+(8v^2-8v^2\rho^2)v^2{y_j}^2),\\
\zeta_{2,4} =&\pm(2v\kappa\theta-2v^2y_j\kappa-2v^3)h^3+2v^3y_jh^2+(1-\mu)k(\pm2(v^3\kappa-\kappa^2\theta v\\
&+\kappa^2v^2y_j)h^3+(4\kappa^2v^2{y_j}^2-(2v^2+8\kappa\theta)\kappa v y_j+2\kappa\theta(2\kappa\theta-v^2)-2v^4)h^2\\
&\pm((-8v^3\kappa+4v^4\rho){y_j}^2+(8\kappa\theta v^2-8v^3\rho r)y_j)h+(-8v^4\rho^2+8v^4){y_j}^2),\\
\zeta_{5,7} =&v^3\rho y_jh^2+(1-\mu)k((v^3{y_j}^2\kappa-v(v\kappa\theta+2r\kappa v+\kappa v^2\rho)y_j\\
&-v(v^2r-2r\kappa\theta+v^3\rho))h^2\pm(-v(2v^3\rho+v^3+4\kappa v^2\rho+2v^2\kappa){y_j}^2\\
&+v(2v\kappa\theta+4v\kappa\theta\rho+4v^2\rho r+2v^2r)y_j)h+v(2v^3+6v^3\rho+4v^3\rho^2){y_j}^2),
\end{aligned}$$

$$\begin{aligned}\zeta_{6,8} = &-v^3\rho y_j h^2 + (1-\mu)k((-v^3{y_j}^2\kappa + v(v\kappa\theta + 2r\kappa v + \kappa v^2\rho)y_j \\ &+ v(v^2r - 2r\kappa\theta + v^3\rho))h^2 \pm (v(-2v^3\rho + v^3 + 4\kappa v^2\rho - 2v^2\kappa){y_j}^2 \\ &+ v(2v\kappa\theta - 4v\kappa\theta\rho + 4v^2\rho r - 2v^2 r)y_j)h + v(2v^3 - 6v^3\rho + 4v^3\rho^2){y_j}^2).\end{aligned}$$

When multiple indexes are used with $\pm$ and $\mp$ signs, the first and second index corresponds to the upper and lower sign, respectively. In the Crank-Nicolson case $\mu = 1/2$, the resulting scheme is of order two in time and of order four in space.

3 Stability results

We study the von Neumann stability of the scheme (for frozen coefficients). Note that our numerical experiments that we reported in [3,4] did not reveal any stability problems. To reduce the high number of parameters, we assume zero interest rate, $r = 0$, and choose the parameter $\mu = 1/2$. We rewrite $u_{i,j}^n$ as

$$u_{i,j}^n = g^n e^{Iiz_1 + Ijz_2}, \tag{4}$$

where I is the imaginary unit, g^n is the amplitude at time level n, and $z_1 = 2\pi h/\lambda_1$ and $z_2 = 2\pi h/\lambda_2$ are phase angles with wavelengths $\lambda_{1,2}$ in the range $[0, 2\pi[$. The scheme is stable if for all $z_{1,2}$ the amplification factor $G = g^{n+1}/g^n$ satisfies

$$|G|^2 - 1 \leq 0. \tag{5}$$

An expression for G can be found using (4) in (3). We recall the following theorem.

Theorem 1 (cf. [3]). *For $r = \rho = 0$ and $\mu = 1/2$ (Crank-Nicolson), the scheme (3) is unconditionally stable (von Neumann).*

One key ingredient of the proof in [3] is to define new variables $c_{1,2} = \cos(z_{1,2}/2)$, $s_{1,2} = \sin(z_{1,2}/2)$, $W = -2(vy - \theta)s_2/v$, $V = 2vys_1/\kappa$, which allow us to express G in terms of h, k, κ, V, W and trigonometric functions only. For non-zero correlation the situation becomes more involved. Additional terms appear in the expression for the amplification factor G and we face an additional degree of freedom through ρ. Since we have proven unconditional stability for $\rho = 0$ it seems reasonable to assume condition (5) to hold at least for values of ρ close to zero. In practical applications, however, correlation can be strongly negative. Few theoretical results can be obtained, some of them are given in the following lemma.

Lemma 1. *For any ρ, $r = 0$, $\mu = 1/2$ (Crank-Nicolson), if $c_1 = \pm 1$ or $c_2 = \pm 1$ or $y = 0$ then the stability condition* (5) *is satisfied.*

Proof. We can prove by direct computation

- if $y = 0$, it holds $|G|^2 - 1 = 0$, and
- if $c_1^2 = 1$ (then $V = 0$), it holds $|G|^2 - 1 = -2$, and
- if $c_2^2 = 1$ (then $W = 0$), it holds $|G|^2 - 1 = -64\alpha\kappa kVh^2 s_1/(\beta_1 s_1 + \beta_2)$,

where $\alpha = h^2(c_1^2-1)-6{c_1}^2-12$, $\beta_1 = 32\kappa kV(c_1^2-1)h^4-192\kappa kV({c_1}^2+2)h^2$, $\beta_2 = 16(c_1^2-1)h^6+(\kappa^2k^2V^2({c_1}^2-1)-64{c_1}^4-256({c_1}^2+1))h^4-12\kappa^2k^2V^2({c_1}^2+2)h^2+144\kappa^2k^2V^2(c_1^2-1)$, $s_1 \in [0,1]$, and $V \geq 0$. It is simple to prove that $\alpha \leq 0$, $\beta_1 \leq 0$, $\beta_2 \leq 0$ and conclude. □

Since at present a complete analysis for non-zero correlation seems out of reach, we resort to performing numerical studies of the amplification factor G. To this end, we fix some parameters to practical relevant values, $v = 0.1$, $\kappa = 2$, $\theta = 0.01$. We replace all sinus terms in (5) by equivalent cosinus expressions. Then, condition (5) depends on ρ and five other parameters: c_1, c_2, y, h, k.

We reformulate condition (5) into a constrained optimization problem with constraints induced by typical parameter ranges: $c_1, c_2 \in [-1,1]$, $y \in [0,2]$, $h \in [10^{-6},10^{-1}]$ and $k \in [10^{-12},10^{-1}]$ (no real restriction on the mesh widths). For different values of ρ fixed in $[-1,0]$, we search

$$\max_{c_1,c_2,y,h,k} |G(\rho)|^2 - 1 \tag{6}$$

which has to be less or equal to zero. A line-search global-optimization algorithm based on the Powell's and Brent's methods [1, 14] is employed. More precisely, we use the DirectSearch optimization package v.2 for Maple [13] and its derivative-free optimisation method CDOS (Conjugate Direction with Orthogonal Shift). Solving (6) for 50 uniform values of $\rho \in [-1,0]$, we find that the stability condition is always satisfied. The maxima for each ρ are always negative and very close to 0. This result is in agreement with Lemma 1 ($|G|^2-1 = 0$ for $y = 0$). We conjecture that the stability condition (5) is satisfied although hard to prove analytically.

Moreover, these results give information on the location of the maxima. We observe that extrema are often attained for y close to 0 as already mentioned, and for the extreme values of $c_{1,2} = \pm 1$ which correspond to vanishing V and W, respectively, and h,k seem to be linked. By Lemma 1 the stability condition is satisfied for such values which induce drastic simplification in G. To study the behavior of G according to ρ away from these values, we solve (6) restricting the range of the parameters to exclude those specific values (where stability is satisfied). We consider $c_{1,2} \in [-1+\varepsilon, 1-\varepsilon]$ with $\varepsilon = 10^{-6}$, $h \in [10^{-6},10^{-1}]$ and fix $k = h^2$ as suggested by the above results and the parabolic nature of the PDE. We split the interval for y into $[\varepsilon, 2/10]$ (to exclude 0 and observe significant maximum values) and $[2/10,2]$ (to exclude $y = \theta/v$ which cancels W). Figure 3 gives the maxima obtained for 50 uniform values of $\rho \in [-1,0]$ and illustrates the influence of ρ. The stability condition is more and more difficult to obtain as $\rho \searrow -1$ or $y \searrow 0$. The stability condition is always satisfied. We refer to [3] for additional numerical experiments where we monitored the error of numerical solutions for vanishing and for non-zero correlation. We observed a similar behavior for both cases and did not observe any stability problems.

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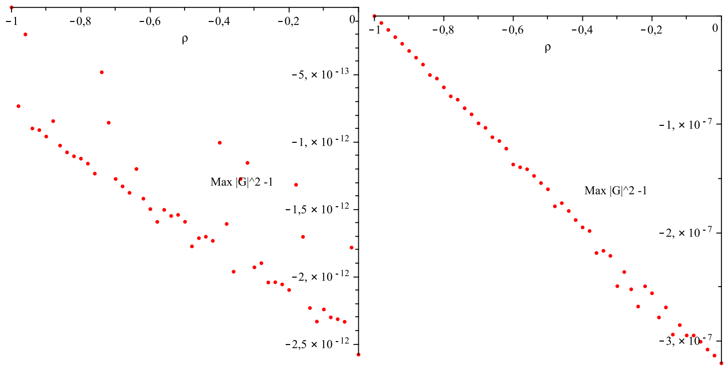

Fig. 1 Numerical results for (6) for $\rho \in [-1,0]$ and $y \in [\varepsilon, 2/10]$ (left), $y \in [2/10, 2]$ (right).

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